## COUNTING PRINCIPLES

2.8. Suppose a bookcase shelf has 5 history texts, 3 sociology texts, 6 anthropology texts, and 4 psychology texts. Find the number $n$ of ways a student can choose: $(a)$ one of the texts; (b) one of each type of text.
(a) Here the sum rule applies; hence $n=5+3+6+4=18$.
(b) Here the product rule applies; hence $n=5 \cdot 3 \cdot 6 \cdot 4=360$.
2.9. A restaurant has a menu with 4 appetizers, 5 entrees, and 2 desserts. Find the number $n$ of ways a customer can order an appetizer, entree, and dessert.

Here the product rule applies since the customer orders one of each. Thus $n=4 \cdot 5 \cdot 2=40$.
2.10. A history class contains 8 male students and 6 female students. Find the number $n$ of ways that the class can elect: (a) 1 class representative; (b) 2 class representatives, 1 male and 1 female; (c) 1 president and 1 vice-president.
(a) Here the sum rule is used; hence $n=8+6=14$.
(b) Here the product rule is used; hence $n=8 \cdot 6=48$.
(c) There are 14 ways to elect the president, and then 13 ways to elect the vice-president. Thus, $n=14 \cdot 13=182$.
2.11. There are 5 bus lines from city $A$ to city $B$ and 4 bus lines from city $B$ to city $C$. Find the number $n$ of ways a person can travel by bus:
(a) from $A$ to $C$ by way of $B,(b)$ round-trip from $A$ to $C$ by way of $B$,
(c) round-trip from $A$ to $C$ by way of $B$, without using a bus line more than once.
(a) There are 5 ways to go from $A$ to $B$ and 4 ways to go from $B$ to $C$; hence, by the product rule, $n=5 \cdot 4=20$.
(b) There are 20 ways to go from $A$ to $C$ by way of $B$ and 20 ways to return. Thus, by the product rule, $n=20 \cdot 20=400$.
(c) The person will travel from $A$ to $B$ to $C$ to $B$ to $A$. Enter these letters with connecting arrows as follows:

$$
A \rightarrow B \rightarrow C \rightarrow B \rightarrow A
$$

There are 5 ways to go from $A$ to $B$ and 4 ways to go from $B$ to $C$. Since a bus line is not to be used more than once, there are only 3 ways to go from $C$ back to $B$ and only 4 ways to go from $B$ back to $A$. Enter these numbers above the corresponding arrows as follows:

$$
A \xrightarrow{-} B \xrightarrow{4} C \xrightarrow{3} B \xrightarrow{4} A
$$

Thus, by the product rule, $n=5 \cdot 4 \cdot 3 \cdot 4=240$.
2.12. Suppose there are 12 married couples at a party. Find the number $n$ of ways of choosing a man and a woman from the party such that the two are: (a) married to each other, (b) not married to each other.
(a) There are 12 married couples and hence there are $n=12$ ways to choose one of the couples.
(b) There are 12 ways to choose, say, one of the men. Once the man is chosen, there are 11 ways to choose the women, anyone other than his wife. Thus, $n=12(11)=132$.
2.13. Suppose a password consists of 4 characters, the first 2 being letters in the (English) alphabet and the last 2 being digits. Find the number $n$ of:
(a) passwords, (b) passwords beginning with a vowel
(a) There are 26 ways to choose each of the first 2 characters and 10 ways to choose each of the last 2 characters. Thus, by the product rule,

$$
n=26 \cdot 26 \cdot 10 \cdot 10=67,600
$$

(b) Here there are only 5 ways to choose the first character. Hence $n=5 \cdot 26 \cdot 10 \cdot 10=13,000$.

## PERMUTATIONS AND ORDERED SAMPLES

2.14. State the essential difference between permutations and combinations, with examples.

Order counts with permutations, such as words, sitting in a row, and electing a president, vice-president, and treasurer. Order does not count with combinations, such as committees and teams (without counting positions). The product rule is usually used with permutations since the choice for each of the ordered positions may be viewed as a sequence of events.
2.15. Find the number $n$ of ways that 4 people can sit in a row of 4 seats.

The 4 empty seats may be pictured by
$\qquad$
, $\qquad$ , $\qquad$ , -
The first seat can be occupied by any one of the 4 people, that is, there are 4 ways to fill the first seat. After the first person sits down, there are only 3 people left and so there are 3 ways to fill the second
seat. Similarly, the third seat can be filled in 2 ways, and the last seat in 1 way. This is pictured by

$$
4,3,2,1
$$

Thus, by the product rule, $n=4 \cdot 3 \cdot 2 \cdot 1=4!=24$.
Alternately, $n$ is the number of permutations of 4 things taken 4 at a time, and so

$$
n=P(4,4)=4!=24
$$

2.16. A family has 3 boys and 2 girls. (a) Find the number of ways they can sit in a row. (b) How many ways are there if the boys and girls are each to sit together?
(a) The 5 children can sit in a row in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=5$ ! $=120$ ways.
(b) There are 2 ways to distribute them according to sex: $B B B G G$ or $G G B B B$. In each case, the boys can sit in $3 \cdot 2 \cdot 1=3!=6$ ways and the girls can sit in $2 \cdot 1=2!=2$ ways. Thus, altogether, there are $2 \cdot 3!\cdot 2!=2 \cdot 6 \cdot 2=24$ ways.
2.17. Find the number $n$ of distinct permutations that can be formed from all the letters of each word:
(a) THOSE, (b) UNUSUAL, (c) SOCIOLOGICAL.

This problem concerns permutations with repetitions.
(a) $n=5!=120$, since there are 5 letters and no repetitions.
(b) $n=\frac{7!}{3!}=840$, since there are 7 letters of which 3 are $U$ and no other letter is repeated.
(c) $n=\frac{12!}{3!2!2!2!}$, since there are 12 letters of which 3 are $O, 2$ are $C, 2$ are $I$, and 2 are $L$.
2.18. Find the number $n$ of different signals, each consisting of 6 flags hung in a vertical line, that can be formed from 4 identical red flags and 2 identical blue flags.

This problem concerns permutations with repetitions. Thus, $n=\frac{6!}{4!2!}=15$ since there are 6 flags of which 4 are red and 2 are blue.
2.19. Find the number $n$ of ways that 7 people can arrange themselves: $(a)$ in a row of 7 chairs, $(b)$ around a circular table.
(a) The 7 people can arrange themselves in a row in $n=7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=7$ ! ways.
(b) One person can sit at any place at the circular table. The other 6 people can then arrange themselves in $n=6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=6$ ! ways around the table.

This is an example of a circular permutation. In general, $n$ objects can be arranged in a circle in $(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1=(n-1)$ ! ways.
2.20. Suppose repetitions are not allowed. (a) Find the number $n$ of three-digit numbers that can be formed from the six digits: $2,3,5,6,7,9$. (b) How many of them are even? (c) How many of them exceed 400 ?

There are 6 digits, and the three-digit number may be pictured by

In each case, write down the number of ways that one can fill each of the positions.
(a) There are 6 ways to fill the first position, 5 ways for the second position, and 3 ways for the third position. This may be pictured by: $\qquad$ 6 5 , 4 . Thus, $n=6 \cdot 5 \cdot 4=120$.
Alternately, $n$ is the number of permutations of 6 things taken 3 at a time, and so

$$
n=P(6,3)=6 \cdot 5 \cdot 4=120
$$

(b) Since the numbers must be even, the last digit must be either 2 or 4 . Thus, the third position is filled first and it can be done in 2 ways. Then there are now 5 ways to fill the middle position and 4 ways to fill the first position. This may be pictured by: $4, \quad, 5, \ldots 2$. Thus, $4 \cdot 5 \cdot 2=120$ of the numbers are even.
(c) Since the numbers must exceed 400 , they must begin with $5,6,7$, or 9 . Thus, we first fill the first position and it'can be done in 4 ways. Then there are 5 ways to fill the second position and 4 ways to fill the third position. This may be pictured by: $4,5,4,4$. Thus, $4 \cdot 5 \cdot 4=80$ of the numbers exceed 400.
2.21. A class contains 8 students. Find the number of ordered samples of size 3 :
(a) with replacement, (b) without replacement.
(a) Each student in the ordered sample can be chosen in 8 ways; hence there are $8 \cdot 8 \cdot 8=8^{3}=512$ samples of size 3 with replacement.
(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $8 \cdot 7 \cdot 6=336$ samples of size 3 without replacement.
2.22. Find $n$ if:
(a) $P(n, 2)=72$,
(b) $2 P(n, 2)+50=P(2 n, 2)$.
(a) $\quad P(n, 2)=n(n-1)=n^{2}-n$; hence

$$
n^{2}-n=72 \quad \text { or } \quad n^{2}-n-72=0 \quad \text { or } \quad(n-9)(n+8)=0
$$

Since $n$ must be positive, the only answer is $n=9$.
(b) $P(n, 2)=n(n-1)=n^{2}-n$ and $P(2 n, 2)=2 n(2 n-1)=4 n^{2}-2 n$. Hence:

$$
\begin{array}{ll}
2\left(n^{2}-n\right)+50=4 n^{2}-2 n & \text { or } 2 n^{2}-2 n+50=4 n^{2}-2 n \\
& \text { or } 50=2 n^{2} \quad \text { or } \quad n^{2}=25
\end{array}
$$

Since $n$ must be positive, the only answer is $n=5$.

## COMBINATIONS AND PARTITIONS

2.23. There are 12 students who are eligible to attend the National Student Association annual meeting. Find the number $n$ of ways a delegation of 4 students can be selected from the 12 eligible students.

This concerns combinations, not permutations, since order does not count in a delegation. There are " 12 choose 4 " such delegations. That is,

$$
n=C(12,4)=\binom{12}{4}=\frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1}=495
$$

2.24. A student is to answer 8 out of 10 questions on an exam.
(a) Find the number $n$ of ways the student can choose the eight questions.
(b) Find $n$ if the student must answer the first three questions.
(a) The 8 questions can be selected " 10 choose 8 " ways. That is,

$$
n=C(10,8)=\binom{10}{8}=\binom{10}{2}=\frac{10 \cdot 9}{2 \cdot 1}=45
$$

(b) If the first 3 questions are answered, then the student must choose the other 5 questions from the remaining 7 questions. Hence

$$
n=C(7,5)=\binom{7}{5}=\binom{7}{2}=\frac{7 \cdot 6}{2 \cdot 1}=21
$$

2.25. A class contains 10 students with 6 men and 4 women. Find the number $n$ of ways:
(a) A 4-member committee can be selected from the students.
(b) A 4-member committee with 2 men and 2 women.
(c) The class can elect a president, vice-president, treasurer, and secretary.
(a) This concerns combinations, not permutations, since order does not count in a committee. There are " 10 choose 4 " such committees. That is,

$$
n=C(10,4)=\binom{10}{4}=\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}=210
$$

(b) The 2 men can be chosen from the 6 men in $\binom{6}{2}$ ways and the 2 women can be chosen from the 4 women in $\binom{4}{2}$ ways. Thus, by the product rule,

$$
n=\binom{6}{2}\binom{4}{2}=\frac{6 \cdot 5}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1}=15(6)=90 \text { ways }
$$

(c) This concerns permutations, not combinations, since order does count. Thus,

$$
n=P(6,4)=6 \cdot 5 \cdot 4 \cdot 3=360
$$

2.26. A box contains 7 blue socks and 5 red socks. Find the number $n$ of ways two socks can be drawn from the box if: (a) They can be any color; $(b)$ They must be the same color.
(a) There are " 12 choose 2 " ways to select 2 of the 12 socks. That is,

$$
n=C(12,2)=\binom{12}{2}=\frac{12 \cdot 11}{2 \cdot 1}=66
$$

(b) There are $C(7,2)=21$ ways to choose 2 of the 7 blue socks and $C(5,2)=10$ ways to choose 2 of the 5 red socks. By the sum rule, $n=21+10=31$.
2.27. Let $A, B, \ldots, L$ be 12 given points in the plane $\mathbb{R}^{2}$ such that no 3 of the points lie on the same line. Find the number $n$ of:
(a) Lines in $\mathbb{R}^{2}$ where each line contains two of the points.
(b) Lines in $\mathbb{R}^{2}$ containing $A$ and one of the other points.
(c) Triangles whose vertices come from the given points.
(d) Triangles whose vertices are $A$ and two of the other points. Since order does not count, this problem involves combinations.
(a) Each pair of points determines a line; hence

$$
n=" 12 \text { choose } 2 "=C(12,2)=\binom{12}{2}=\frac{12 \cdot 11}{2 \cdot 1}=66
$$

(b) We need only choose one of the 11 remaining points; hence $n=11$.
(c) Each triple of points determines a triangle; hence

$$
n=" 12 \text { choose } 3 "=C(12,3)=\binom{12}{3}=220
$$

(d) We need only choose two of the 11 remaining points; hence $n=C(11,2)=55$. (Alternately, there are $C(11,3)=165$ triangles without $A$ as a vertex; hence $220-165=55$ of the triangles do have $A$ as a vertex.)
2.28. There are 12 students in a class: Find the number $n$ of ways that 12 students can take 3 different tests if 4 students are to take each test.

There are $C(12,4)=495$ ways to choose 4 students to take the first test; following this, there are $C(8,4)=70$ ways to choose 4 students to take the second test. The remaining students take the third test. Thus

$$
n=70(495)=34,650
$$

2.29. Find the number $n$ of ways 12 students can be partitioned into 3 teams $A_{1}, A_{2}, A_{3}$, so that each team contains 4 students. (Compare with the preceding Problem 2.28.)

Let $A$ denote one of the students. There are $C(11,3)=165$ ways to choose 3 other students to be on the same team as $A$. Now let $B$ be a student who is not on the same team as $A$. Then there are $C(7,3)=35$ ways to choose 3 from the remaining students to be on the same team as $B$. The remaining 4 students form the third team. Thus, $n=35(165)=5925$.

Alternately, each partition $\left[A_{1}, A_{2}, A_{3}\right]$ can be arranged in $3!=6$ ways as an ordered partition. By the preceding Problem 2.28, there are 34,650 such ordered partitions. Thus, $n=34,650 / 6=5925$.
2.30. Find the number $n$ of committees of 5 with a given chairperson that can be selected from 12 persons.

Method 1: The chairperson can be chosen in 12 ways and, following this, the other 4 on the committee can be chosen from the remaining 11 people in $C(11,4)=330$ ways. Thus,

$$
n=12(330)=3960
$$

Method 2: The 5 -member committee can be chosen from the 12 persons in $C(12,5)=792$ ways. Each committee can then select a chairman in 5 ways. Thus,

$$
n=5(792)=3960
$$

2.31. There are $n$ married couples at a party. (a) Find the number $N$ of (unordered) pairs at the party. (b) Suppose every person shakes hands with every other person other than his or her spouse. Find the number $M$ of handshakes.
(a) There are $2 n$ people at the party, and so there are " $2 n$ choose 2 " pairs. That is,

$$
N=C(2 n, 2)=\frac{2 n(2 n-1)}{2}=n(2 n-1)=2 n^{2}-n
$$

(b) $M$ is equal to the number of pairs who are not married. There are $n$ married pairs. Thus, using (a),

$$
M=2 n^{2}-n-n=2 n^{2}-2 n=2 n(n-1)
$$

## TREE DIAGRAMS

2.32. Construct the tree diagram that gives the permutations of $\{a, b, c\}$.

The tree diagram, drawn downward with the "root" on the top, appears in Fig. 2-5. Each path from the root to an endpoint ("leaf") of the tree represents a permutation. There are 6 such paths which yield the following 6 permutations:

$$
a b c, a c b, b a c, b c a, c a b, c b a
$$



Fig. 2-5
2.33. Audrey has time to play roulette at most 5 times. At each play she wins or loses $\$ 1$. She begins with $\$ 1$ and will stop playing before 5 plays if she loses all her money.
(a) Find the number of ways the betting can occur.
(b) How many cases will she stop before playing 5 times?
(c) How many cases will she leave without any money?

Construct the appropriate tree diagram as shown in Fig. 2-6. Each number in the diagram denotes the number of dollars she has at that moment in time. Thus, the root, which is circled, is labeled with the number 1.


Fig. 2-6
(a) There are 14 paths from the root of the tree to an endpoint ("leaf"), so the betting can occur in 14 different ways.
(b) There are only 2 paths with less than 5 edges, so Audrey will not play 5 times in only 2 of the cases.
(c) There are 4 paths which end in 0 or, in other words, only 4 of the leaves are labeled with 0 . Thus, Audrey will leave without any money in 4 of the cases.
$!$

